Substituting these in (1),

$$\frac{d^2(2\omega t - \theta)}{dt^2} = -\omega^2 \sin(2\omega t - \theta), \tag{2}$$

or, since $2\omega t - \theta = \varphi$,

$$\frac{d^2\varphi}{dt^2} = -\omega^2 \sin \varphi. \tag{3}$$

Multiplying by $2(d\varphi/dt)$ and integrating,

$$\frac{d\varphi^2}{dt^2} = 2\omega^2 \cos\varphi + C. \tag{4}$$

When $\varphi = 0$, $d\varphi/dt = 0$; hence, $C = 2\omega^2$, and (4) is

$$\frac{d\varphi^2}{dt^2} = 2\omega^2(1 + \cos\varphi) = 4\omega^2 \cos^2 \tfrac{1}{2}\varphi. \tag{5}$$

This gives

$$\omega dt = \frac{d\frac{1}{2}\varphi}{\cos\frac{1}{2}\varphi}; \tag{6}$$

integrating and using the fact that $\varphi = 0$ when $t = 0$, we have

$$\log\tan\frac{\pi + \varphi}{4} = \omega t. \tag{7}$$

Also solved by G. PAASWELL and J. B. REYNOLDS.

346 (Mechanics). Proposed by WILLIAM HOOVER, Columbus, Ohio.

Half the length of one of the equal parts of a uniform heavy string resting in equilibrium over a smooth horizontal indefinitely thin peg is cut off; determine the instantaneous change in the pressure on the peg.

SOLUTION BY THE PROPOSER.

Let m = the mass of a unit of length of the string, $2a$ = the whole length of the string, and so a, $a/2$, the parts at the instant of cutting, $3a/2$ the length in motion after any time t from the beginning of motion, x = the longer part at the same instant, and T = the corresponding tension in the string; then the equations of motion are, noticing that the momenta of the moving masses are each variable,

$$\frac{d}{dt}\left(mx\frac{dx}{dt}\right) = m\left(\frac{dx^2}{dt^2} + x\frac{d^2x}{dt^2}\right) = mgx - T, \tag{1}$$

$$\frac{d}{dt}\left\{m(\tfrac{3}{2}a - x)\frac{d}{dt}(\tfrac{3}{2}a - x)\right\} = \frac{d}{dt}\left\{-m\tfrac{3}{2}a\frac{dx}{dt} + mx\frac{dx}{dt}\right\} = mg(\tfrac{3}{2}a - x) - T. \tag{2}$$

Subtracting (2) from (1), we have

$$\frac{d^2x}{dt^2} = \frac{2g}{3a}\left(2x - \frac{3a}{2}\right). \tag{3}$$

Multiplying by $2(dx/dt)$ and integrating, noticing that when $x = a$, $(dx/dt) = 0$, we obtain

$$\frac{dx^2}{dt^2} = \frac{g}{3a}\left\{\left(2x - \frac{3a}{2}\right)^2 - \frac{a^2}{4}\right\}. \tag{4}$$

Substituting the values of dx^2/dt^2 and d^2x/dt^2 in (1), and putting $T = T_0$ when $x = a$, we find $T_0 = \frac{2}{3}mga$.

If P = the required initial pressure, $P = 2T_0 = \frac{4}{3}mga = 2mga - \frac{1}{3}\cdot 2mga$, so that the pressure before cutting is diminished by one third.

Also solved by HORACE OLSON.

264 (Number Theory). Proposed by C. F. GUMMER, Kingston, Canada.

Find a general formula for three squares in arithmetical progression. Is it possible for the common difference to be a perfect square?

SOLUTION BY ARTEMAS MARTIN, LL.D., Washington, D. C.

Let x^2, y^2, z^2 be three square numbers in arithmetical progression; then we must have

$$y^2 - x^2 = z^2 - y^2, \qquad \text{or} \qquad x^2 + z^2 = 2y^2. \tag{1}$$

Assume $z = v + w$, $x = v - w$, and (1) becomes after dividing by 2,

$$v^2 + w^2 = y^2. \tag{2}$$

Take now $v = p^2 - q^2$, $w = 2pq$, and (2) is satisfied. Retracing, we find

$$z = p^2 - q^2 + 2pq, \qquad x = p^2 - q^2 - 2pq, \qquad y = p^2 + q^2.$$

Hence, the required squares are

$$x^2 = (p^2 - q^2 - 2pq)^2, \qquad y^2 = (p^2 + q^2)^2, \qquad z^2 = (p^2 - q^2 + 2pq)^2.$$

Taking $p = 2$, $q = 1$, the numbers are 1, 25, 49; taking $p = 3$, $q = 2$, the numbers are 49, 169, 289; taking $p = 4$, $q = 1$, the numbers are 49, 289, 529; taking $p = 4$, $q = 3$, the numbers are 289, 625, 961; and so on, indefinitely.

The common difference of three square numbers in arithmetical progression can not be a square number. See Barlow's "Theory of Numbers," p. 257. An equivalent theorem is also given in Carmichael's *Diophantine Analysis*, p. 14.

There can not be *four* square numbers in arithmetical progression. Barlow, same page. Therefore there can not be *five*, nor any greater number than three, of squares in arithmetical progression.

Also solved by J. L. RILEY, V. M. SPUNAR, and the PROPOSER.

265 (Number Theory). Proposed by J. W. NICHOLSON, Louisiana State University.

If the roots of $x^4 - ax^2 + bx + c = 0$ are rational, prove that $4(a + yz) - 3(y + z)^2$ is a perfect square, y and z being any two roots of the equation.

SOLUTION BY N. P. PANDYA, Sojitra, India.

Since y and z are roots of the given equation, $x^2 - x(y + z) + yz$ is a factor of the left-hand side of the equation.

Since the term in x^3 is wanting, the remaining roots are given by

$$x^2 + x(y + z) + \frac{c}{yz} = 0. \tag{1}$$

The product of (1) with $x^2 - x(y + z) + yz = 0$ gives

$$a = -\frac{c}{yz} - yz + (y + z)^2.$$

Hence,

$$4(a + yz) = -\frac{4c}{yz} + 4(y + z)^2,$$

or

$$4(a + yz) - 3(y + z)^2 = (y + z)^2 - \frac{4c}{yz} = \text{a square},$$

since the roots of (1) are rational and its discriminant is therefore a square.

267 (Number Theory). Proposed by C. C. YEN, Tangshan, North China.

A number theory function $\phi(n)$ is defined for every positive integer n, and for every such number n it satisfies the relation $\phi(d_1) + \phi(d_2) + \phi(d_3) + \cdots + \phi(d_r) = n$, where $d_1, d_2, \cdots, d_r$ are the divisors of n. From this property alone show that

$$\phi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_k}\right),$$

where p p $p_3 \cdots p_k$ are the different prime factors of n.